2.1 – Trigonometry review

Recall that the three most useful trigonometric ratios are relationships between the lengths of the sides in a right triangle as defined by the following table:

<table>
<thead>
<tr>
<th>Full ratio name</th>
<th>Standard abbreviation</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine(A)</td>
<td>sin(A)</td>
<td>( \frac{\text{side opposite } A}{\text{hypotenuse}} ) commonly referred to as ( \frac{O}{H} )</td>
</tr>
<tr>
<td>cosine(A)</td>
<td>cos(A)</td>
<td>( \frac{\text{side adjacent to } A}{\text{hypotenuse}} ) commonly referred to as ( \frac{A}{H} )</td>
</tr>
<tr>
<td>tangent(A)</td>
<td>tan(A)</td>
<td>( \frac{\text{side opposite } A}{\text{side adjacent to } A} ) commonly referred to as ( \frac{O}{A} )</td>
</tr>
</tbody>
</table>

These ratios are used to find unknown values in a right triangle as in the following example:

Example 1

Find sinA, cosA, and tanA in the following triangle.
Solution:

Since the hypotenuse is needed to find sin and cos, that is calculated first using the Pythagorean Theorem:

\[ h = \sqrt{13.3^2 + 21.5^2} = 25.3 \text{m} \]

The three trigonometric ratios can now be calculated:

\[
\sin A = \frac{O}{H} = \frac{13.3}{25.3} = 0.5257
\]

\[
\cos A = \frac{A}{H} = \frac{21.5}{25.3} = 0.8498
\]

\[
\tan A = \frac{O}{A} = \frac{13.3}{21.5} = 0.6186
\]

A scientific calculator (such as the one provided with the Windows operating system) can be used to calculate values associated with trigonometric values as well. For example, the cos of a 60° angle is 0.5 as indicated by the following expression:

\[ \cos 60° = 0.5 \]
Different applications will require that the value of 0.5 is found given that the angle is 60°. Other applications will demand that the angle measurement of 60° is found given that the cos value is 0.5. In either case, a scientific calculator will be able to calculate the solution if used correctly.

Example 2a

Find $\sin 33.2^\circ$

Solution:

33.2 is entered into the calculator, the ‘sin’ button is pressed, and the value is displayed giving:

$\sin 33.2^\circ = 0.5476$

Example 2b

Find $A$ given that $\tan A = 2.713$

Solution:

2.713 is entered into the calculator, the ‘Inv’ button is pressed followed by the ‘$\tan^{-1}$’ button and the value is displayed giving:

If $\tan A = 2.713$, then $A = 69.77^\circ$
2.2 – Vectors

Typical measurements that provide only one piece of data are referred to as scalar measurements. For example, if a doctor must weigh a patient and the scale reads 187 lbs., the single scalar number 187 provides all of the necessary information.

Vectors are measurements that incorporate two distinct pieces of information typically referred to as the magnitude and direction. Vectors are particularly important in technological applications as they are fundamental to electronic circuitry, physical force applications, manufacturing processes, and much more.

The following provides a specific example as to when a vector is needed. If it is necessary to know the affect a wind will have on the flight of a plane, it is clear that the wind speed is an essential piece of data. However, if the pilot is told that the wind speed is 20 kph, this will not help to determine the overall affect. It is also necessary to know from which direction the wind is blowing in order to determine its overall affect. Hence, it would be far more useful to tell the pilot that the wind is blowing 20 kph @ 6° West of North. In this case, the magnitude is 20 kph and the direction is 6° West of North. This will provide the pilot with all of the necessary information (note that today there is technology that can sense and automatically correct for wind speed, but this technology must be programmed by people who understand the mathematics).

Vectors are represented mathematically as arrows. The length of the arrow is indicative of the vector’s magnitude, while its orientation is indicative of direction. However, in a vector system all of these measurements are relative (and hence meaningless until there is more than one vector in the system).

Consider the following vector $v$:

![Vector v]

This vector clearly has a length and a direction, but since there are no other vectors in the system these measurements have no real value.

Now consider the following two vectors $v$ and $w$:

![Vectors v and w]
While there are still no hard and fast values, it can now be determined that relative to each other, \( w \) has a larger magnitude (since the arrow is longer) and we can compare the two directions as needed.

**Vector arithmetic – addition and subtraction**

Many systems contain multiple vector entities and it is often necessary to add or subtract them to determine the overall effect. The most straightforward method to add vectors is the triangle method. The following two step algorithm provides the means to do this:

1. Redraw the vectors so that the tail of the second vector (i.e. the end without the arrow) is adjacent to the head of the first (i.e. the end with the arrow).

2. Draw a third vector that starts at the tail of the first and ends at the head of the second. This third vector represents the sum and is called the resultant (and often notated with the letter \( R \)).

**Example 1**

Add the vectors \( v \) and \( w \) above to find \( v + w \).

**Solution:**

1. The two vectors are drawn with the head of \( v \) adjacent to the tail of \( w \):

   ![Diagram](image1.png)

2. The resultant vector is drawn to find the sum of \( v \) and \( w \):

   ![Diagram](image2.png)
So in this system, adding \( v \) and \( w \) results in the vector \( R \). In other words, \( v + w = R \).

This process can be applied to systems with more than 2 vectors. Each subsequent vector is placed in the same way until all vectors have been placed tail-to-head. Finally, the resultant is drawn starting with the tail of the first vector and ending at the head of the last. This process is demonstrated in the following example.

**Example 2**

Add the following four vectors.

\[ \vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4} \]

Solution:

1. The vectors are placed tail-to-head. The order they are placed is inconsequential as the resultant will be the same regardless.
2. The resultant vector is drawn to find the sum of these 4 vectors:

So in this system, adding \( V_1, V_2, V_3, \) and \( V_4 \) results in the vector \( R \). In other words, \( V_1 + V_2 + V_3 + V_4 = R \).

Subtracting vectors is based on vector addition. To subtract one vector from another, the opposite of the vector being subtracted is added to the first vector. The opposite of a vector is defined to be the vector with the same magnitude but opposite direction as in the following image:

![Diagram of vector addition and subtraction](image)

Combining the concepts of vector addition and opposite vectors, one vector can be subtracted from another as in the following example:

**Example 3**

Find \( V_1 - V_2 \).
Solution:

1. \( V_1 - V_2 \) can be rewritten as \( V_1 + (-V_2) \). Hence the tail of \(-V_2\) is placed adjacent to the head of \( V_1\):

2. The resultant vector is drawn to find the difference of these vectors:

So in this system, \( R \) is the difference. In other words, \( V_1 - V_2 = R \).

2.3 – Vector applications

There are many technology problems that can be resolved by combining the trigonometric concepts and the vector concepts in the previous two sections. These include (but are not limited to) simple force systems, simply aviation problems, police investigations, and electronics applications. The following examples address some of these ideas.

Example 1
Two forces at right angles to each other are acting on an object. One is pushing to the right with a force of 50 lbs., and the other is pushing up with a force of 72 lbs. What is the net force on the object?

Solution:

1. Let \( f_1 \) be the force pushing to the right, and \( f_2 \) be the force pushing up. Now the following two forces are both pushing on the object:

2. Since both of these forces are acting on the object, they must be added together in order to find the net effect on the object:

3. The resulting image can now be used to find the magnitude of the vector (i.e. its length) and its direction (i.e. the angle \( B \)).

   a. The magnitude is found using the Pythagorean Theorem:

   \[
   | R | = \sqrt{50^2 + 72^2} = 87.7 \text{ lb}
   \]

   (Note that the magnitude is represented using two vertical bars around the vector name)
b. Angle B can be found if angle A is known. Angle A can be found using trigonometry:

\[
\tan A = \frac{50}{72} = 0.6944 \\
A = 34.8^\circ
\]

Since A and B are complements of each other, \( B = 90^\circ - A = 90^\circ - 34.8^\circ \)
\( B = 55.2^\circ \)

Putting together the information found above, it is deduced that the net force on the object is represented by the vector 87.7 lbs. @ 55.2°.
Example 2

A plane is flying due east at 317 mph. The wind is blowing due south at 26 mph. Accounting for the effect of the wind, what is the true speed and direction of the plane relative to the ground?

Solution:

1. The system below demonstrates the vectors involved, with R being the true direction and speed of the plane.

\[ \text{P = 317 mph} \quad \text{R} \quad \text{W = 26 mph} \]

2. The resulting image can now be used to find the magnitude of the vector (i.e. its length) and its direction (i.e. the marked angle between P and R).
   
   a. The magnitude of R represents the speed and is found using the Pythagorean Theorem:
      \[ |R| = \sqrt{317^2 + 26^2} = 318 \text{ mph} \]

   b. The angle between P and R (referred to as A below) can be found using trigonometry:
      \[ \tan A = \frac{26}{317} = 0.0820 \]
      \[ A = 4.7^\circ \]

Putting together the information found above, it is deduced that the true direction and speed of the plane is: 318 mph @ 4.7° south of east.

However, it is more typical to see these types of measurements written this way:

318 mph @ 85.3° east of south (it is standard to state either north or south as the last direction).
2.4 – Vector components

A pair of vectors form the components of a third vector if the third vector represents their sum. For example, the red vectors $V_1$ and $V_2$ are components of the black vector $V$ in the following system:

While all vectors have an infinite number of component pairs, the most useful component pair is that which contains one horizontal and one vertical vector. These are commonly referred to as $V_x$ for the horizontal component and $V_y$ for the vertical. For the same vector $W$, the horizontal and vertical components would be those in the figure below:

Since $V_x$ and $V_y$ always form a right angle, trigonometry can be used to derive formulas as follows if the vector $V$ is known.
Finding the components of a vector when the vector is known.

Formulas to find the components of a vector if the vector is known

| \[ V_x = |V| \cos \theta \]  
| \[ V_y = |V| \sin \theta \]  

Example 1

If \( V = 101 \text{ ft/s} \circ 72^\circ \), find \( V_x \) and \( V_y \)

Solution:

\[ V_x = |V| \cos \theta = 101 \cos(72) = 31.2 \text{ ft/s} \]  
\[ V_y = |V| \sin \theta = 101 \sin(72) = 96.1 \text{ ft/s} \]  

So \( V_x = 31.2 \text{ ft/s} \) and \( V_y = 96.1 \text{ ft/s} \)

Notice that even though the components are vectors, the direction is left off because \( V_x \) is always at \( 0^\circ \) and \( V_y \) is always at \( 90^\circ \).
Example 2

In a series circuit, the impedance (Z), phase angle (\( \Phi \)), inductive reactance (\( X_L \)), and the resistance (R) are all key pieces of information (Z, XL, and R are all measured in Ohms notated by the symbol \( \Omega \)).

The following image demonstrates these relationships visually:

![Triangle Diagram](image)

In a series circuit, if \( Z = 23\Omega \) and \( \phi \) measures 35° find \( X_L \) and R.

Solution:
R and \( X_L \) and R represent the horizontal and vertical components, and so can be found using the corresponding formulas.

\[
R = |Z| \cos \phi = 23 \cos(35^\circ) = 18.8\Omega
\]

\[
X_L = |Z| \sin \phi = 23 \sin(35^\circ) = 13.2\Omega
\]
Finding the vector when the components are known.

In the above examples, the components of a vector were found when the vector was known. Sometimes the reverse process is necessary. The formulas below provide a way to determine the original vector if only the components are known.

**Formulas to find a vector when its components are known**

|V| = the magnitude of V, and θ is the direction of V

\[ |V| = \sqrt{V_x^2 + V_y^2} \]

\[ \theta = \tan \left( \frac{V_y}{V_x} \right) \]

**Example 3**

If \( V_x = 68.1 \text{ kph} \) and \( V_y = 89.3 \text{ kph} \), find \( V \)

**Solution:**

\[ |V| = \sqrt{V_x^2 + V_y^2} = \sqrt{68.1^2 + 89.3^2} = 112.3 \text{ kph} \]

\[ \tan \theta = \frac{V_y}{V_x} = \frac{89.3}{68.1} \text{ and so } \theta = 52.7^\circ \]

So \( V = 112.3 \text{ kph @ 52.7}^\circ \)
Example 4

In a series circuit, if \( X_L = 15\Omega \) and \( R = 18\Omega \), find \( Z \) and \( \phi \).

Solution:
The same image as in the above example illustrates the relationships:

Since \( R \) and \( X_L \) represent the horizontal and vertical components, \( Z \) and \( \phi \) can be found using the corresponding formulas:

\[
Z = \sqrt{15^2 + 18^2} = 23.4\Omega
\]

\[
\tan \phi = \frac{15}{18} \text{ and so } \phi = 39.8^\circ
\]

So \( Z \) (the impedance) = 23.4\( \Omega \) and \( \phi \) (the phase angle) = 39.8\(^\circ\)
Chapter 2 Homework problems

Section 2.1 Homework

Calculate the following.

1. Find \( \sin A, \cos A, \tan A, \sin B, \cos B, \) and \( \tan B \) in the following triangle:

2. Find \( \sin A, \cos A, \tan A, \sin B, \cos B, \) and \( \tan B \) in the following triangle:

3. If \( A = 62.7^\circ \), find:
   a. \( \sin A \)
   b. \( \cos A \)
   c. \( \tan A \)

4. If \( B = 119^\circ \), find:
   a. \( \sin B \)
   b. \( \cos B \)
   c. \( \tan B \)
5. Find A if:
   a. \( \sin A = 0.6212 \)
   b. \( \sin A = 3.215 \)
   c. \( \cos A = 0.1144 \)
   d. \( \tan A = 2.511 \)

6. If \( \cos A = 0.3519 \), find \( \tan A \)

**Section 2.2 Homework**

Use the following vectors to answer the questions below.

![Vectors diagram]

1. Construct \( A + B \)
2. Construct \( B + C \)
3. Construct \( B + C + D \)
4. Construct \( A + B + A \)
5. Construct \( A - B \)
6. Construct \( A + C - D \)

**Section 2.3 Homework**

1. Two forces at right angles to each other are acting on an object. One is pushing to the right with a force of 38.7 lbs., and one is pushing up with a force of 18.4 lbs. What is the net force on the object?

2. Answer question number one assuming the first force is pushing to the left instead of the right.
3. A plane is flying due east at 422 mph. The wind is blowing due south at 31 mph. Accounting for the effect of the wind, what is the true speed and direction of the plane relative to the ground?

4. A plane is flying due north 412 mph. The wind is blowing due east at 20 mph. Accounting for the effect of the wind, what is the true speed and direction of the plane relative to the ground?

5. Answer question number 4 assuming the wind is blowing due south.

Section 2.4 Homework

For each of the following vectors in questions 1 thru 4, find the components $V_x$ and $V_y$.

1. $V$: 30 mph @ 72°

2. $V$: 41.3 m/s @ 110°

3. $V$: 800 lb. @ 204°

4. $V$: 102 ft. @ 314°

5. Explain the positive or negative signs on the components in the problems above.

For each of the following component pairs in question 6 thru 9, find the corresponding vector.

6. $V_x = 20 \text{ mph}$
   $V_y = 12 \text{ mph}$

7. $V_x = -160 \text{ m/s}$
   $V_y = 88 \text{ m/s}$

8. $V_x = -0.3 \text{ lb}$
   $V_y = -0.215 \text{ lb}$

9. $V_x = 17.2 \text{ ft}$
   $V_y = -76.1 \text{ ft}$

10. Are the positive or negative signs on the components in the problems above consistent with your findings in questions 1 – 5?
11. In a series circuit, if \( Z = 15\Omega \) and \( \phi = 17^\circ \), find \( X_L \) and \( R \).

12. In a series circuit, if \( X_L = 20\Omega \) and \( R = 22\Omega \), find \( Z \) and \( \phi \).

13. Challenge question: This scenario does not have right angles, but vector components can be applied strategically so that right angles can be formed and trigonometry can be applied.

Two forces are acting on an object. One is pushing the object with a force of 17 lbs. at 34°. The other is pushing with a force of 28 lbs. at 82°. What is the net force on the object?